ZAPREMINA

90. Naći zapreminu tijela koje nastaje rotacijom figure ograničene krivom $y = 2x - x^2$ i pravom $y = 0$ oko: a) Ox ose, b) Oy ose

$y=0$ → x osa

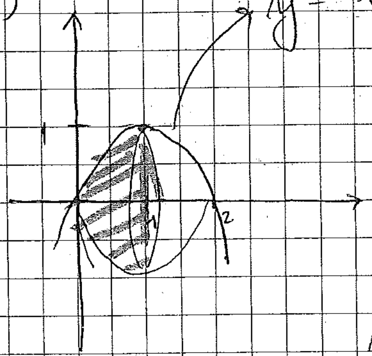
$y = -x^2 + 2x$

$y = 0 \iff x(2-x) = 0 \iff x = 0 \vee x = 2 \rightarrow O(0,0)\ A(2,0)$

$a=-1;\ b=2;\ c=0;\ D=4;$ Tjeme $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

$T\left(-\frac{2}{-2}, -\frac{4}{-4}\right) \longrightarrow T(1,1)$

a)



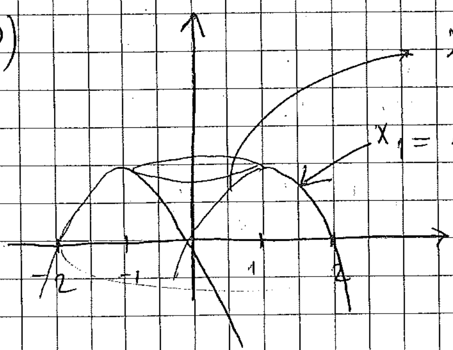
$V = 2\pi \int_0^1 y^2(x)\,dx$

$V = 2\pi \int_0^1 (2x - x^2)^2\,dx =$

$= 2\pi \int_0^1 (4x^2 - 4x^3 + x^4)\,dx =$

$= \frac{16}{15}\pi \longrightarrow$ a more i $V = \pi \int_0^2 y^2(x)\,dx$

b)



$x_2(y) = 1 - \sqrt{1-y}$

$x_1 = 1 + \sqrt{1-y}$

$y = 2x - x^2$

$x^2 - 2x + y = 0$

$x_{1,2} = \frac{2 \pm \sqrt{4-4y}}{2}$

$x_{1,2} = 1 \pm \sqrt{1}$

$x_{1,2} = \frac{2 \pm 2\sqrt{1-y}}{2}$

$x_{1,2} = 1 \pm \sqrt{1-y}$

$$V = \pi \int_0^1 \left(x_1^2(y) - x_2^2(y)\right) dy$$

$$V = \pi \int_0^1 \left((1+\sqrt{1-y})^2 - (1-\sqrt{1-y})^2\right) dy$$

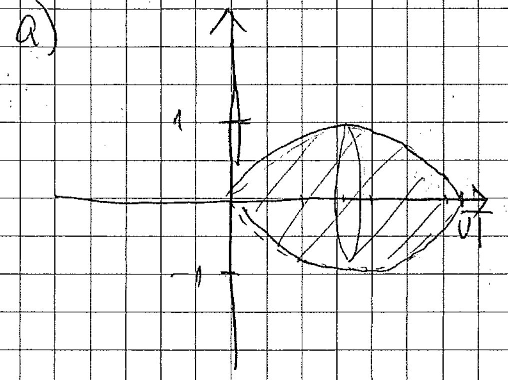
$$V = \pi \int_0^1 \left(\cancel{1} + 2\sqrt{1-y} + \cancel{1} \cancel{-y} \cancel{-1} + 2\sqrt{1-y} \cancel{-1+y}\right) dy =$$

$$= \pi \cdot 4 \int_0^1 \sqrt{1-y}\,dy = \left| \begin{matrix} 1-y=t \\ dy=-dt \end{matrix} \quad \begin{matrix} y & 0 & 1 \\ t & 1 & 0 \end{matrix} \right| = \frac{8\pi}{3}$$

$$= -4\pi \int_1^0 \sqrt{t}\,dt = 4\pi \int_0^1 \sqrt{t}\,dt = 4\pi \cdot \frac{2}{3}\sqrt{t^3}\Big|_0^1$$

91) Naći zapreminu tijela koje nastaje rotacijom krive $y = \sin x$ i prave $y=0$ na segmentu $[0, \pi]$ oko a) Ox-ose; b) Oy ose

a)



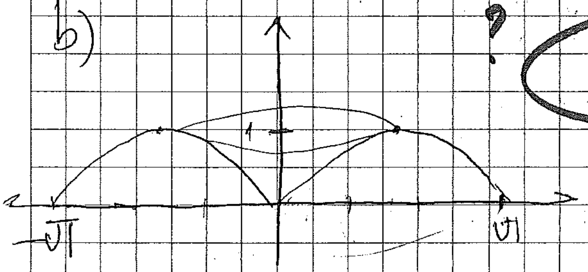
$$V = \pi \int_0^{\pi} \sin^2 x \, dx = \pi \int_0^{\pi} \frac{1-\cos 2x}{2} dx =$$

$$= \frac{\pi}{2} x \Big|_0^{\pi} - \frac{\pi}{2} \cdot \frac{1}{2} \sin 2x \Big|_0^{\pi} =$$

$$= \frac{\pi^2}{2}$$

$$V_y = 2\pi \int_a^b x f(x) \, dx$$

b)



?

$$\boxed{V_y = \pi \int_0^{\pi} x \sin x \, dx} = \text{parcijalno} = \dots =$$

$$= \pi\left(-x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x \, dx\right)$$

$$= \pi\left(\pi + \sin x \Big|_0^{\pi}\right) = \pi^2$$

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92) U tački $P(3, y_0)$ parabole $y^2 = 2(x-1)$ povučena je tangenta. Izračunati zapreminu tijela koje nastaje rotacijom oko Ox ose, figure ograničene tangentom, parabolom i Ox osom.

$y^2 = 2x - 2$ → $a = \frac{1}{2}$; $b = 0$; $c = 1$; $D < 0$ → nema

$2x = y^2 + 2$ presjeka sa y osom

$x = \frac{1}{2} y^2 + 1$ $a > 0 \Rightarrow$ desno, ~~otvorena~~

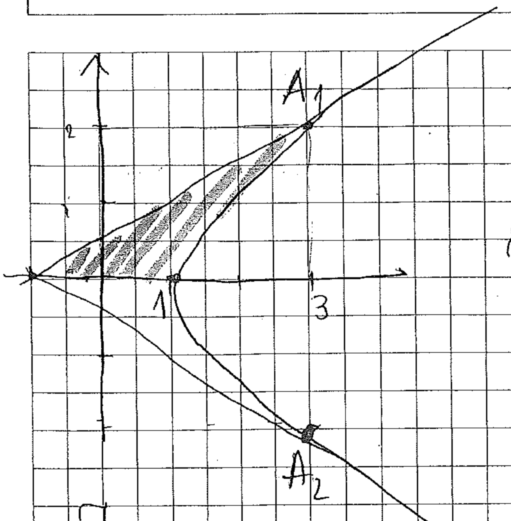
tjeme $\left(-\frac{D}{4a}, -\frac{b}{2a}\right) \to T\left(-\frac{-2}{2}, 0\right) \to T(1, 0)$ → presjek sa Ox osom $y = 0$

P pripada paraboli

$y_0^2 = 2(3-1)$ $A_1(3, 2)$ → Zbog simetričnosti

$y_0^2 = 4 \to y_0 = \pm 2$ $A_2(3, -2)$ dovoljna tangenta u jednoj tački

JORDI LABANDA



Tangenta u tački $A_1(3,2)$

$y-2 = y'(3)(x-3)$

$y^2 = 2x-2$

$2yy' = 2 \to y' = \frac{1}{y}$

$y'(A_1) = \frac{1}{2}$

J-na tangente $\to y-2 = \frac{1}{2}(x-3)$: t_1

$t_1: \; y = \frac{1}{2}x + \frac{1}{2}$ $\qquad t_2 \to$ simetrično

→ Zapremina ista kao kad rotira pola il cijela

→ zapremina je razlika 2 zapremine:

$V = V_1 - V_2$

$$V = \pi \int_1^3 \left(\frac{1}{2}x + \frac{1}{2}\right)^2 dx - \pi \int_1^3 (2x-2)\,dx = \frac{4}{3}\pi$$

93) Izračunati zapreminu nastalu rotacijom figure ograničene krivom $x^2+y^2=16$ i krivom $y^2=6x$:

a) oko Ox-ose

b) oko Oy-ose

J-na kruga sa centrom u $O(0,0)$ i poluprečnikom 4

* Presjek kruga i parabole

$$\begin{cases} x^2+y^2=16 \\ y^2 = 6x \end{cases} \to x^2+6x-16=0$$

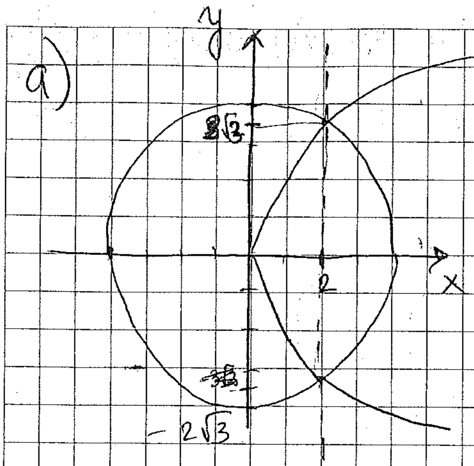
$$x_{1,2} = \frac{-6 \pm \sqrt{36+64}}{2} = \quad x_1 = \frac{-6+10}{2} = \boxed{2}, \quad \cancel{x_2 = -8}$$

$\boxed{y^2 \geq 0}$ $\qquad y^2 = 6\cdot x$, $x \geq 0$

$x_1 = 2 \to y = \pm\sqrt{12} = \pm 2\sqrt{3}$

$A_1(2, 2\sqrt{3}) \qquad A_2(2, -2\sqrt{3})$

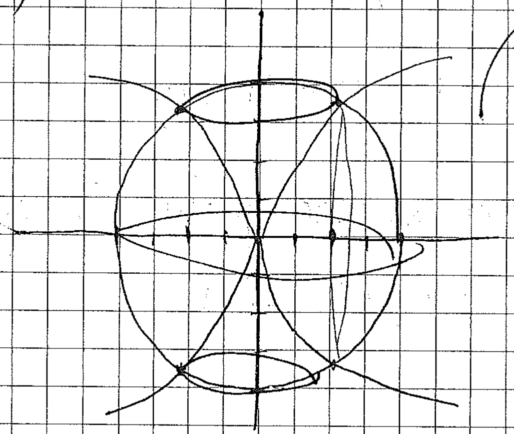
a)



$$V=\pi\int_0^2 6x\,dx+\pi\int_2^4\left(16-x^2\right)dx=$$

$$=\frac{76}{3}\pi$$

b)



$$x^2+y^2=16 \to x^2=16-y^2$$

$$y^2=6x \to x=\frac{1}{6}y^2 \to x^2=\frac{1}{36}y^4$$

dio kruga   parabola

$$V=\pi\int_0^{2\sqrt3}\left(16-y^2-\frac{1}{36}y^4\right)dy\cdot 2$$

$$\cancel{V=25}\quad V=\frac{224}{5}\sqrt3\,\pi$$

$$V=\pi\left(16y-\frac{y^3}{3}-\frac{1}{36}\cdot\frac{1}{5}y^5\right)\Big|_0^{2\sqrt3}=$$

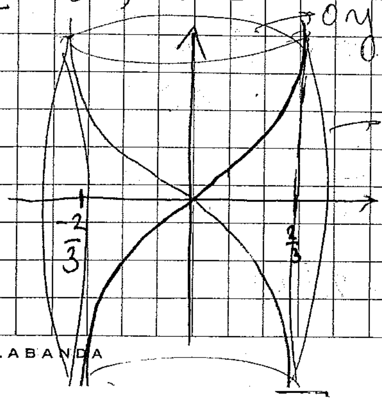
$$=\pi\left(16\cdot2\sqrt3-\frac{2^3\cdot3\cdot\sqrt3}{3}-\frac{1}{36}\cdot\frac{1}{5}\cdot\frac{2^5\cdot3^2\sqrt3}{}\right)=$$

$$32-2^3-\frac{32\cdot9}{36\cdot5}=\frac{8}{9}\cdot\frac{9}{5}=\frac{8}{5}$$

$$\frac{112}{5}\cdot2\to\frac{224}{5}\sqrt3\,\pi \qquad \frac{8}{5}$$

94 POVRŠINA ROTACIONOG TIJELA

94 Luk krive $y=x^3$ rotira oko Ox ose na segmentu $\left[-\frac{2}{3},\frac{2}{3}\right]$. Naći površinu rotacionog tijela



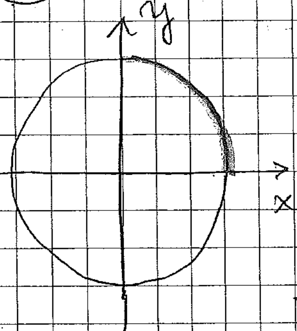
$$P=2\cdot2\pi\int_0^{2/3}y(x)\sqrt{1+y'^2(x)}\,dx,$$

$$y(x)=x^3;\quad y'(x)=3x^2$$

$$P = 4\pi \int_0^{\frac{2}{3}} x^3 \cdot \sqrt{1+9x^4}\, dx = \left[\begin{array}{l} 1+9x^4 = t \\ x^3\, dx = \frac{1}{36}\, dt \end{array} \quad \begin{array}{c|c|c} x & 0 & \frac{2}{3} \\ \hline t & 1 & \frac{25}{9} \end{array} \right] =$$

$$= 4\pi \cdot \frac{1}{36} \cdot \int_1^{\frac{25}{9}} \sqrt{t}\, dt = \frac{2\pi}{27} \sqrt{t^3}\Big|_1^{\frac{25}{9}} = \frac{196}{729}\pi$$

(95) Površina lopte poluprečnika r



$$x^2+y^2=r^2 \rightarrow y^2 = r^2-x^2 \rightarrow y = \pm\sqrt{R^2-x^2}$$

lopta nastaje rotacijom ovog kruga oko Ox ose

→ svejedno da li rotira cio ili pola kruga → neka rotira $\frac{1}{4}$ pa množimo sa 2

$$P = 2 \cdot 2\pi \int_0^r y(x) \cdot \sqrt{1+y'^2(x)}\, dx$$

$$y(x) = \sqrt{r^2-x^2} \qquad y = \sqrt{R^2-x^2}$$

↓ jer uzimamo + dio

$$y'(x) = -\frac{x}{\sqrt{r^2-x^2}}$$

$$P = 4\pi \int_0^r \sqrt{r^2-x^2} \cdot \sqrt{1+\frac{x^2}{r^2-x^2}}\, dx =$$

$$= 4\pi \int_0^r \cancel{\sqrt{r^2-x^2}} \cdot \frac{r}{\cancel{\sqrt{r^2-x^2}}}\, dx$$

$$P = 4\pi r \int_0^r dx = 4\pi r\, x\Big|_0^r = 4\pi r^2 = P$$

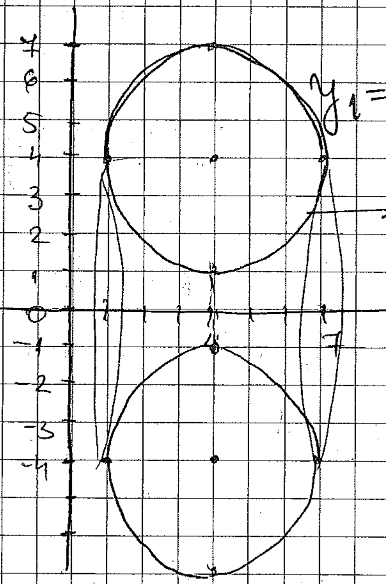
Kombinovano → kolokvijumi

96. Izračunati površinu i zapreminu tijela koje se dobija rotacijom krive $x^2+y^2-8x-8y+23=0$ oko Ox ose.

$x^2+y^2-8x-8y+23=0 \to (x-4)^2+(y-4)^2=9$

$(x-4)^2+(y-4)^2-9=0$

Jednačina kruga sa centrom u tački $A(4,4)$ i poluprečnikom 3



$y_1=4+\sqrt{9-(x-4)^2}$

$y_2=4-\sqrt{9-(x-4)^2}$

$$V=\pi\int_1^7 \left(y_1^2(x)-y_2^2(x)\right)dx=$$

$$=\pi\int_1^7\left(\left(4+\sqrt{9-(x-4)^2}\right)^2-\left(4-\sqrt{9-(x-4)^2}\right)^2\right)dx=$$

$$=\pi\int_1^7\left(\cancel{16}+8\sqrt{9-(x-4)^2}+\cancel{9-(x-4)^2}-\cancel{16}+8\sqrt{9-(x-4)^2}-\left(\cancel{9-(x-4)^2}\right)\right)dx$$

$$=\pi\int_1^7 16\sqrt{9-(x-4)^2}\,dx=\left[\begin{array}{l} x-4=t \\ dx=dt \end{array}\;\begin{array}{c|c|c} x & 1 & 7 \\ \hline t & -3 & 3 \end{array}\right]=$$

$$=16\pi\int_{-3}^{3}\sqrt{9-t^2}\,dt=\left[\begin{array}{l} t=3\sin z \\ dt=3\cos z\,dz \\ z=\arcsin\frac{t}{3} \end{array}\;\begin{array}{c|c|c} t & -3 & 3 \\ \hline z & -\frac{\pi}{2} & \frac{\pi}{2} \end{array}\right]=$$

$$=16\pi\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}3\cdot 3\sqrt{1-\sin^2 z}\,\cos z\,dz=144\pi\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 z\,dz=$$

$\frac{1}{2}\left(\cos\frac{\pi}{2}-\cos\right.$

$$=144\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2z}{2}\,dz=144\pi\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{2}\;\; 72\pi^2$$

Povrsina → gornji dio rotira + donji dio rotira

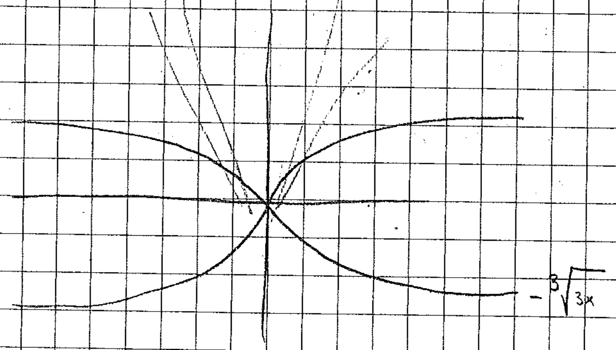
$$P = 2\pi \int_1^7 y_1(x) \cdot \sqrt{1+y_1'^2(x)}\, dx + 2\pi \int_1^7 y_2(x) \sqrt{1+y_2'^2(x)}\, dx$$

$$y_1'(x) = \frac{-(x-4)}{\sqrt{9-(x-4)^2}} \quad ; \quad y_2'(x) = \frac{x-4}{\sqrt{9-(x-4)^2}}$$

$P_1 =$

97) Figura F ograničena krivama $y=-\sqrt[3]{3x}$, $y=3|x|$ rotira oko Oy ose. Skicirati figuru F i izračunati površinu tijela koje nastaje rotacijom.

$$y=3|x| = \begin{cases} -3x, & x<0 \\ 3x, & x\geq 0\end{cases}$$



→ presjek krive i prave

→ $y=-\sqrt[3]{3x}$, $y=3|x|$

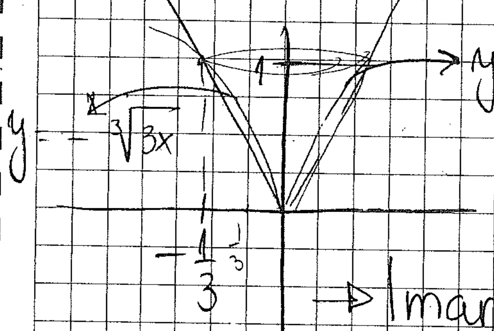
$$-\sqrt[3]{3x} = 3|x| \,/^3 \to -3x = 3^3\cdot|x^3|$$

$$-x = 9|x^3| \,/:x$$

$$+1 = 9|x^2|$$

$$x^2=\frac{1}{9} \to x = \pm\frac{1}{3}$$

ali



→ Ako je fja sa kojom radimo negativna → !! apsolutna vr.

→ Imamo nešto što opisuje kriva i nešto što opisuje prava → $P=P_1+P_2$

$y=\sqrt[3]{3x}$

$y^3=3x$

$x=\frac{1}{3}y^3$

$x'=\frac{1}{\cancel{3}}\cancel{3}y^2$

$$P_1=\int_0^1 x(y)\cdot\sqrt{1+x'^2(y)}\,dy$$

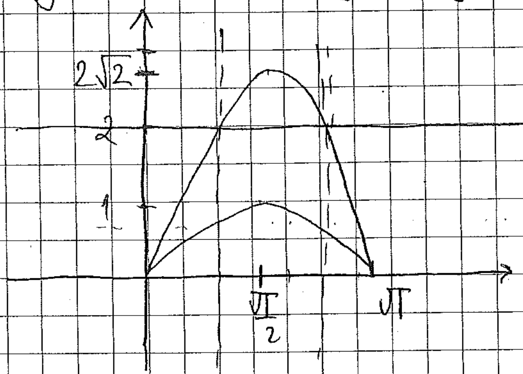
$$P_1=\int_0^1 \frac{1}{3}y^3\cdot\sqrt{1+y^4}\,dy = \frac{1}{3}\int_0^1 y^3\sqrt{1+y^4}\,dy =$$

$1+y^4=t$

$3y^3dy=dt$

$y^3dy=\frac{1}{3}dt$

(98) Figura ograničena lukovima krivih $y = \sin x$ i $y = 2\sqrt{2}\sin x$ na intervalu $[0, \pi]$ i pravom $y = 2$ rotira oko Ox ose. Izračunati zapreminu tako dobijenog rotacionog tijela.



→ presjek

dovršiti kući :D

NESVOJSTVENI INTEGRALI

99. a) $\int_0^{+\infty} x \cdot e^{-2x} dx = \lim_{B\to+\infty} \int_0^B x e^{-2x} dx = \left[\begin{array}{l} x = U \to dU = dx \\ V = \int e^{-2x} dx = -\frac{1}{2} e^{-2x} \end{array} \right]$

$= -\frac{1}{2} \lim_{B\to+\infty} \left(x e^{-2x} \Big|_0^B - \int_0^b e^{-2x} dx \right) =$

$= -\frac{1}{2} \lim_{B\to+\infty} \left(\underbrace{\frac{x}{e^{2x}} \Big|_0^B}_{0} + \frac{1}{2} \left(\underbrace{\frac{1}{e^{2B}}}_{0} - 1 \right) \right) = \frac{1}{4}$

$0 \to e^{2x} \to$ brže raste od $y = x$

b) $\int_1^{+\infty} \frac{dx}{x\sqrt{1+x^2}} = \lim_{B\to+\infty} \int_1^B \frac{dx}{x\sqrt{1+x^2}} = \left[\begin{array}{l} \frac{1}{x} = t \to x = \frac{1}{t} \\ \to dx = -\frac{1}{t^2} \\ x^2 = \frac{1}{t^2} \\ \to \begin{array}{c|c|c} x & 1 & B \\ t & 1 & \frac{1}{B} \end{array} \end{array} \right]$

$= -\lim_{B\to+\infty} \int_1^{\frac{1}{B}} \frac{\frac{1}{t^2}}{\frac{1}{t} \sqrt{1+\frac{1}{t^2}}} dt =$

$= -\lim_{B\to+\infty} \int_1^{\frac{1}{B}} \frac{dt}{\sqrt{t^2+1}} = -\lim_{B\to+\infty} \left(\ln \left| t + \sqrt{t^2+1} \right| \right) \Big|_1^{\frac{1}{B}} =$

$= -\lim_{B\to+\infty} \left(\underbrace{\ln \left| \frac{1}{B} + \sqrt{\frac{1}{B^2} + 1} \right|}_{\ln 1 = 0} - \ln \left| 1 + \sqrt{1+1} \right| \right) =$

$= \cancel{-\lim_{B\to+\infty}} \quad = \ln(1+\sqrt{2})$

100. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{B\to1^-} \int_0^B \frac{dx}{\sqrt{1-x^2}} = \lim_{B\to1^-} \arcsin x \Big|_0^B =$

$= \lim_{B\to1^-} (\arcsin B - \underbrace{\arcsin 0}_{0}) = \frac{\pi}{2}$

101. $$\int_2^3 \frac{dx}{\sqrt{6x-x^2-8}} = \left[\begin{array}{l} D = 36-32 = 4 \to \\ x_{1,2} = \frac{-6\pm 2}{-2} \begin{array}{l} \nearrow 4 \\ \searrow 2 \end{array} \\ -(x-2)(x-4) \end{array} \right] =$$

$$= \int_2^3 \frac{dx}{\sqrt{-(x-2)(x-4)}} = \lim_{A\to 2^+} \int_a^3 \frac{dx}{\sqrt{-(x-2)(x-4)}} = \left[1-(x-3)^2 \right]$$

$$= \lim_{a\to 2^+} \int_a^3 \frac{dx}{\sqrt{1-(x-3)^2}} = \lim_{a\to 2^+} \left(\arcsin(x-3)\Big|_a^3 \right) =$$

$$= \lim_{a\to 2^+} \left(\underbrace{\cancel{\arcsin 0}}_{0} - \arcsin(a-3) \right) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

102. problem u tački 0 → zato nesvojstveni

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt[3]{\sin^2 x}}\,dx = \int_{-\frac{\pi}{2}}^{0} \frac{\cos x}{\sqrt[3]{\sin^2 x}}\,dx + \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt[3]{\sin^2 x}}\,dx =$$

$$= \lim_{B\to 0^-} \int_{-\frac{\pi}{2}}^{B} \frac{\cos x}{\sqrt[3]{\sin^2 x}}\,dx + \lim_{A\to 0^+} \int_A^{\frac{\pi}{2}} \frac{\cos x}{\sqrt[3]{\sin^2 x}}\,dx = \left[\begin{array}{l} \sin x = t \\ \cos x\,dx = dt \end{array} \right] =$$

$$\sin^{-\frac{2}{3}} x$$

$$= \lim_{B\to 0^-} 3\sqrt[3]{\sin x}\Big|_{-\frac{\pi}{2}}^{B} + \lim_{A\to 0^+} 3\sqrt[3]{\sin x}\Big|_A^{\frac{\pi}{2}} =$$

$$= -(-3) + 3 = 6$$

103. Ispitati konvergenciju integrala

$$\int_0^1 \frac{dx}{x^{\alpha}}$$

$\alpha \neq 1$: $\displaystyle\int_0^1 \frac{dx}{x^\alpha} = \lim_{a\to 0^+} \int_a^1 x^{-\alpha}\,dx = \lim_{a\to 0^+} \frac{x^{-\alpha+1}}{1-\alpha}\Bigg|_a^1 =$

$$= \lim_{a\to 0^+}\left(\frac{1}{1-\alpha} - \frac{a^{1-\alpha}}{1-\alpha}\right) = \begin{cases} +\infty, & \alpha>1 \\ \dfrac{1}{1-\alpha}, & \alpha<1 \end{cases}$$

$\alpha>1 \to 1-\alpha<0 \to a^{\alpha-1}$

$\frac{1}{0} \to \infty$

$\alpha<1 \to 1-\alpha>0 \to \dfrac{\underset{0}{a}^{1-\alpha}}{1-\alpha} \to 0 \to \dfrac{1}{1-\alpha}$

$\alpha=1$: $\displaystyle\int_0^1 \frac{dx}{x} = \lim_{a\to 0^+}\int_a^1 \frac{dx}{x} = \lim_{a\to 0^+} \ln|x|\Big|_a^1 =$

$$= \lim_{a\to 0^+}\left(\underset{0}{\ln 1} - \underset{-\infty}{\ln a}\right) = +\infty$$

→ Za $\alpha<1$ → Integral konvergira i njegova vrijednost je $\frac{1}{1-\alpha}$, a za $\alpha\geq 1$ integral diverg.

(104.) Ispitati konvergenciju

$$\int_1^{+\infty} \frac{dx}{x^\alpha}$$

$\alpha\neq 1$: $\displaystyle\int_1^{+\infty}\frac{dx}{x^\alpha} = \lim_{B\to+\infty} \frac{x^{1-\alpha}}{1-\alpha}\Bigg|_1^B = \lim_{B\to+\infty}\left(\frac{B^{1-\alpha}}{1-\alpha} - \frac{1^{1-\alpha}}{1-\alpha}\right)$

$$= \begin{cases} -\dfrac{1}{1-\alpha}, & \alpha>1 \to \alpha>1 \Rightarrow 1-\alpha<1 \to \dfrac{1}{B^{\alpha-1}} \xrightarrow{B\to\infty} 0 \\ +\infty, & \alpha<1 \to \alpha<1 \Rightarrow 1-\alpha>1 \xrightarrow{B\to+\infty} B^{1-\alpha} \xrightarrow{B\to\infty} \infty \end{cases}$$

$\alpha=1$ $\displaystyle\int_1^{+\infty}\frac{dx}{x^\alpha} = \lim_{B\to\infty}\int_1^B \frac{dx}{x} = \lim_{B\to\infty} \ln x\Big|_1^B =$

$$= \lim_{B\to+\infty}\left(\underset{\infty}{\ln B} - \underset{0}{\ln 1}\right) = \infty$$

→ Za $\alpha>1$ Integral konverg, a za $\alpha\leq 1$ diverg.

105. Ispitati konvergenciju integrala

$$\int_{0}^{+\infty} \frac{\sin x \cdot \operatorname{arctg} x}{1+x^2}\,dx$$

Neka je $f(x) = \dfrac{\sin x \cdot \operatorname{arctg} x}{1+x^2}$

$$|f(x)| = \left| \frac{\sin x \cdot \operatorname{arctg} x}{1+x^2} \right| \le \frac{1 \cdot \frac{\pi}{2}}{1+x^2}, \quad \forall x \ge 0$$

$$|f(x)| \le g(x), \quad g(x) = \frac{\pi}{2} \cdot \frac{1}{1+x^2}$$

$$\int_{0}^{+\infty} g(x)\,dx = \int_{0}^{+\infty} \frac{\pi}{2} \cdot \frac{1}{1+x^2}\,dx = \frac{\pi}{2} \lim_{B \to +\infty} \int_{0}^{B} \frac{dx}{1+x^2} =$$

$$= \frac{\pi}{2} \lim_{B \to +\infty} (\operatorname{arctg} B - \operatorname{arctg} 0) = \frac{\pi^2}{4}$$

$$\Rightarrow \int_{0}^{+\infty} g(x)\,dx \text{ konvergira}$$

$$\left. \begin{array}{l} |f(x)| \le g(x), \quad \forall x \ge 0 \\ \int_{0}^{+\infty} g(x)\,dx \text{ konvergira} \end{array} \right\} \Rightarrow \int_{0}^{+\infty} f(x)\,dx \text{ takođe konvergira}$$